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On mixed Hodge structure of character varieties

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Abstract

In this poster, we study a variety, called character variety $\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,\mathrm{GL}(n,\mathbb{C})}$, whose points parametrize representation of the fundamental group of k-punctured Riemann surface of genus g into $\mathrm{GL}(n,\mathbb{C})$. In particular, we investigate the mixed Hodge structure of character varieties. There exists very interesting conjectures. For example,

Conjecture 1.2.1 of [1]

We have the following
(i) The compactly supported mixed Hodge polynomial $H^*_c(\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,\mathrm{GL}(n,\mathbb{C})};x,y,t)$ is a polynomial in xy and t, and is independent of the choice of generic eigenvalues of multiplicities $\boldsymbol{\mu}$.
(ii) Moreover,

$$H^*_c(\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,\mathrm{GL}(n,\mathbb{C})};x,y,t)=(t\sqrt{q})^{\dim(\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,\mathrm{GL}(n,\mathbb{C})})}\mathbb{H}_{\boldsymbol{\mu}}(-\frac{1}{\sqrt{q}},t\sqrt{q}),$$

where $q:=xy$, and $\mathbb{H}_{\boldsymbol{\mu}}(z,w)$ is the rational function defined in [1].

Here

$$H^*_c(X;x,y,t):=\sum \dim_{\mathbb{C}}(\mathrm{Gr}^F_p\mathrm{Gr}^W_{p+q}H^j_c(X)^{\mathbb{C}})x^py^qt^j.$$

The main result

Conjecture 1.2.1 (i) is true.

λ-connection

Definition

Let Σ be a compact Riemann surface of genus g.

Fix

We fix
- k-distinct points $p_1,\dots,p_k$ in Σ,
- $\boldsymbol{\mu}=(\mu^1,\dots,\mu^k)$, where $\mu^i=(\mu^i_1,\dots,\mu^i_{r_i})$ such that $\mu^i_1\geq\mu^i_2\geq\cdots$ are non-negative integers and $\sum_j\mu^i_j=n$ for each i,
- integers d,n with $n>0$.

We put $\Sigma_0=\Sigma\setminus\{p_1,\dots,p_k\}$, and

$$\Xi^{\boldsymbol{\mu},k}_n(d):=\left\{\left(\lambda,(\xi^i_j)^{1\leq i\leq k}_{1\leq j\leq r_i}\right)\in\mathbb{C}\times\mathbb{C}^{\sum r_i}\;\middle|\;\lambda d+\sum_{i,j}\mu^i_j\xi^i_j=0\right\}.$$

We take a member $(\lambda,\boldsymbol{\xi})\in\Xi^{\boldsymbol{\mu},k}_n(d)$, where $\boldsymbol{\xi}=(\xi^i_j)^{1\leq i\leq k}_{1\leq j\leq r_i}$.

Parabolic λ-connection

We say $(E,\nabla,\{l^{(i)}_*\}_{1\leq i\leq k})$ a $\boldsymbol{\xi}$-parabolic λ-connection of rank n and degree d of type $\boldsymbol{\mu}$ if
(1) E is an algebraic vector bundle on Σ of rank n and degree d,
(2) $\nabla:E\to E\otimes\Omega^1_\Sigma(p_1+\cdots+p_k)$ is a λ-connection, that is, ∇ is a homomorphism of sheaves satisfying $\nabla(fa)=\lambda a\otimes df+f\nabla(a)$ for $f\in\mathcal{O}_\Sigma$ and $a\in E$, and
(3) for each p_i, $l^{(i)}_*$ is a filtration $E|_{p_i}=l^{(i)}_1\supset l^{(i)}_2\supset\cdots\supset l^{(i)}_{r_i}\supset l^{(i)}_{r_i+1}=0$ such that $\dim(l^{(i)}_j/l^{(i)}_{j+1})=\mu^i_j$ and $(\mathrm{Res}_{p_i}(\nabla)-\xi^i_j\mathrm{id}_{E|_{p_i}})(l^{(i)}_j)\subset l^{(i)}_{j+1}$ for $j=1,\dots,r_i$.

Moduli space

Moduli space

There exists a relative coarse moduli scheme

$$\pi:\mathcal{M}^{\boldsymbol{\mu},g,d}_{Hod,\mathrm{GL}(n,\mathbb{C})}\longrightarrow\Xi^{\boldsymbol{\mu},k}_n(d)$$
$$(E,\nabla,\{l^{(i)}_*\}_{1\leq i\leq k})\mapsto(\lambda,\boldsymbol{\xi})$$

of $\boldsymbol{\alpha}$-stable $\boldsymbol{\xi}$-parabolic λ-connections of rank r and degree d of type $\boldsymbol{\mu}$. Moreover, π is smooth and $\mathcal{M}^{\boldsymbol{\mu},g,d}_{Hod,\mathrm{GL}(n,\mathbb{C})}$ is nonsingular.

We denote the fiber of $(\lambda,\boldsymbol{\xi})$ by $\mathcal{M}^{\boldsymbol{\mu},\lambda,\boldsymbol{\xi},g,d}_{Hod,\mathrm{GL}(n,\mathbb{C})}$. In particular, we put $\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\xi},g,d}_{DR,\mathrm{GL}(n,\mathbb{C})}$ and $\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\xi},g,d}_{Dol,\mathrm{GL}(n,\mathbb{C})}$ in the case of $\lambda=1$ and $\lambda=0$ respectively.

Compactification of the moduli space

There exists a natural $\mathbb{C}^\times$-action on $\mathcal{M}^{\boldsymbol{\mu},g,d}_{Hod,\mathrm{GL}(n,\mathbb{C})}$,

$$t\cdot(E,\nabla,\{l^{(i)}_*\})=(E,t\nabla,\{l^{(i)}_*\}).$$

The following $\mathbb{C}^\times$-action on $\Xi^{\boldsymbol{\mu},k}_n(d)$ is well-defined,

$$t\cdot(\lambda,\boldsymbol{\xi})=(t\lambda,t\boldsymbol{\xi}).$$

Then, $\pi:\mathcal{M}^{\boldsymbol{\mu},g,d}_{Hod,\mathrm{GL}(n,\mathbb{C})}\to\Xi^{\boldsymbol{\mu},k}_n(d)$ is a $\mathbb{C}^\times$-equivariant morphism.
Let $\mathcal{M}'$ be the base change of $\mathcal{M}^{\boldsymbol{\mu},g,d}_{Hod,\mathrm{GL}(n,\mathbb{C})}$ via $\mathbb{C}\times\Xi^{\boldsymbol{\mu},k}_n(d)\to\Xi^{\boldsymbol{\mu},k}_n(d)$, given by $(x,(\lambda,\boldsymbol{\xi}))\mapsto(x\lambda,x\boldsymbol{\xi})$. Here the $\mathbb{C}^\times$-action on $\mathbb{C}\times\Xi^{\boldsymbol{\mu},k}_n(d)$ is given by $t\cdot(x,(\lambda,\boldsymbol{\xi}))=(tx,(\lambda,\boldsymbol{\xi}))$. Then, the set $U\subset\mathcal{M}'$ of points $u\in U$ such that $\lim_{t\to\infty}t\cdot(E,\nabla,\{l^{(i)}_*\})$ does not exist is open, and there exists a geometric quotient $\overline{\mathcal{M}}:=U//\mathbb{C}^\times$ which is proper over $\Xi^{\boldsymbol{\mu},k}_n(d)$ via the induce map $\bar\pi:\overline{\mathcal{M}}\to\Xi^{\boldsymbol{\mu},k}_n(d)$.

Theorem 1

The induce map $\bar\pi:\overline{\mathcal{M}}\to\Xi^{\boldsymbol{\mu},k}_n(d)$ is topologically trivial. Moreover any two fibers of π have isomorphic cohomology, and the mixed Hodge structure of the cohomology is pure.

Sketch of proof.

$$\begin{array}{ccc} U/\mathbb{R}^\times_+ & \xrightarrow{\text{(ii)}} & \Xi^{\boldsymbol{\mu},k}_n(d) \\ {\scriptstyle\text{(i)}}\downarrow & & \| \\ \overline{\mathcal{M}} & \xrightarrow{\bar\pi} & \Xi^{\boldsymbol{\mu},k}_n(d), \end{array}$$

where (i) is a U(1)-bundle, and (ii) is topologically trivial. Then, we obtain $\bar\pi$ is topologically trivial.
Note that

$$\overline{\mathcal{M}}\setminus\mathcal{M}=\{\mathbb{C}^\times((E,\nabla,\{l^{(i)}_*\}),0,(\lambda,\boldsymbol{\xi}))\mid\lim_{t\to\infty}t\cdot(E,\nabla,\{l^{(i)}_*\})\text{ exists }\}$$

is trivial over $\Xi^{\boldsymbol{\mu},k}_n(d)$. Then we can show that any fibers of π have isomorphic cohomology; in particular, $H^*(\mathcal{M}^{\boldsymbol{\mu},\lambda,\boldsymbol{\xi},g,d}_{Hod,\mathrm{GL}(n,\mathbb{C})})\cong H^*(\mathcal{M}^{\boldsymbol{\mu},\mathbf{0},g,d}_{Dol,\mathrm{GL}(n,\mathbb{C})})$ for all $(\lambda,\boldsymbol{\xi})\in\Xi^{\boldsymbol{\mu},k}_n(d)$.
Since $\mathcal{M}^{\boldsymbol{\mu},\mathbf{0},g,d}_{Dol,\mathrm{GL}(n,\mathbb{C})}$ is a proper orbifold (in particular a rational homology manifold), its cohomology has pure mixed Hodge structure. By standard Morse theory arguments, $H^*(\mathcal{M}^{\boldsymbol{\mu},\mathbf{0},g,d}_{Dol,\mathrm{GL}(n,\mathbb{C})})\to H^*(\mathcal{M}^{\boldsymbol{\mu},\mathbf{0},g,d}_{Dol,\mathrm{GL}(n,\mathbb{C})})$ is surjective. Thus, $H^*(\mathcal{M}^{\boldsymbol{\mu},\mathbf{0},g,d}_{Dol,\mathrm{GL}(n,\mathbb{C})})$ also has pure mixed Hodge structure. □

Character variety

We now construct a variety, called character variety, whose points parametrize representation of the fundamental group of k-punctured Riemann surface of genus g into $\mathrm{GL}(n,\mathbb{C})$ with prescribed images in semi-simple conjugacy classes $\mathcal{C}_1,\dots,\mathcal{C}_k$ at the puncture. Assume that

$$\prod_{i=1}^k\det\mathcal{C}_i=1$$

and that $(\mathcal{C}_1,\dots,\mathcal{C}_k)$ has type $\boldsymbol{\mu}=(\mu^1,\dots,\mu^k)$; that is, $\mathcal{C}_i$ has type μ^i for each $i=1,\dots,k$, where the type of a semi-simple conjugacy class $\mathcal{C}_i\subset\mathrm{GL}(n,\mathbb{C})$ is defined as the partition $\mu^i=(\mu^i_1,\dots,\mu^i_{r_i})$ describing the multiplicities of the eigenvalues of any matrix in $\mathcal{C}_i$.

Character Variety

For a k-tuple of conjugacy classes $(\mathcal{C}_1,\dots,\mathcal{C}_k)$ of type $\boldsymbol{\mu}$,

$$\mathcal{U}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{\mathrm{GL}(n,\mathbb{C})}:=\{(A_1,B_1,\dots,A_g,B_g,X_1,\dots,X_k)\in\mathrm{GL}(n,\mathbb{C})^{2g}\times\prod_{i=1}^k\mathcal{C}_i$$
$$\mid[A_1,B_1]\cdots[A_g,B_g]X_1\cdots X_k=Id\}.$$

We call the affine GIT quotient by conjugation

$$\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,\mathrm{GL}(n,\mathbb{C})}:=\mathcal{U}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{\mathrm{GL}(n,\mathbb{C})}//\mathrm{PGL}(n,\mathbb{C})=\mathrm{Spec}(\mathbb{C}[\mathcal{U}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{\mathrm{GL}(n,\mathbb{C})}]^{\mathrm{PGL}(n,\mathbb{C})})$$

a character variety of type $\boldsymbol{\mu}$.

If $(\mathcal{C}_1,\dots,\mathcal{C}_k)$ is a generic k-tuple of semi-simple conjugacy classes in $\mathrm{GL}(n,\mathbb{C})$ of type $\boldsymbol{\mu}$, then the quotient $\pi_{\boldsymbol{\mu}}:\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,\mathrm{GL}(n,\mathbb{C})}\to\mathcal{U}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{\mathrm{GL}(n,\mathbb{C})}$ is a principal $\mathrm{PGL}(n,\mathbb{C})$-bundle. Consequently, when nonempty, the affine variety $\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,\mathrm{GL}(n,\mathbb{C})}$ is nonsingular.

Riemann-Hilbert correspondence

For each member $(E,\nabla,\{l^{(i)}_j\})\in\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\xi},g,d}_{DR,\mathrm{GL}(n,\mathbb{C})}$, $\mathrm{Ker}(\nabla^{an}|_{\Sigma_0})$ becomes a local system on Σ_0, where ∇^{an} means the analytic connection corresponding to ∇. The local system $\mathrm{Ker}(\nabla^{an}|_{\Sigma_0})$ corresponds to a representation of $\pi_1(\Sigma_0)$. Then, we obtain an analytic isomorphism,

$$\mathbf{RH}_{\boldsymbol{\xi}}:\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\xi},g,d}_{DR,\mathrm{GL}(n,\mathbb{C})}\to\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g,d}_{B,\mathrm{GL}(n,\mathbb{C})},$$

where $\lambda^i_j=\exp(-2\pi\sqrt{-1}\xi^i_j)$ and we take ξ generic, that is, parabolic connections and local systems are irreducible.
We consider the family $\{\mathbf{RH}_{\boldsymbol{\xi}}\}$ over the locus of generic elements in $\Xi^{\boldsymbol{\mu},\lambda=1,k}_n(d)$, then we obtain the following theorem by Theorem 1,

Theorem 2

The mixed Hodge structure of the cohomology of $\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,\mathrm{GL}(n,\mathbb{C})}$ is independent of the choice of generic eigenvalues of multiplicities $\boldsymbol{\mu}$.

Mixed Hodge structure of character varieties

We consider the cohomology of $\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,\mathrm{GL}(n,\mathbb{C})}$. Since $\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,\mathrm{GL}(n,\mathbb{C})}\cong(\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,\mathrm{SL}(n,\mathbb{C})}\times(\mathbb{C}^\times)^{2g})//\nu_r^{2g}$, where $\nu_n=\{(e^{2\pi\sqrt{-1}d/n}I_n,e^{-2\pi\sqrt{-1}d/n})\mid d=1,\dots,n\}$, we obtain that

$$H^*(\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,\mathrm{GL}(n,\mathbb{C})})\cong H^*(\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,\mathrm{SL}(n,\mathbb{C})})^{\nu_n^{2g}}\otimes H^*((\mathbb{C}^\times)^{2g}).$$

Construction of generators of $H^*(\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,\mathrm{SL}(n,\mathbb{C})})^{\nu_n^{2g}}$

(I) $\mathrm{PGL}(n,\mathbb{C})$-principal bundle on $\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,\mathrm{SL}(n,\mathbb{C})}\times\Sigma_0$:

$$\mathbb{U}:=(\mathrm{PGL}(n,\mathbb{C})\times\mathcal{U}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{\mathrm{SL}(n,\mathbb{C})}\times\tilde\Sigma_0)/(\pi_1(\Sigma_0)\times\mathrm{GL}(n,\mathbb{C})),$$

where $\tilde\Sigma_0$ is the universal covering of Σ_0 and the action is $(p,g)\cdot(h,\rho,x)=(\bar g\rho(p)h,\bar g^{-1}\rho\bar g,p\cdot x)$. The characteristic classes:

$$\bar c_i(\mathbb{U})=\beta_i+\sum_{j=1}^{2g}\gamma_{i,j}e_j+\sum_{k=1}^n\delta_{i,k}f_k\ (i=2,\dots,n),$$

$\beta_i\in H^{2i}(\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,\mathrm{SL}(n,\mathbb{C})})$, $\gamma_{i,j},\delta_{i,k}\in H^{2i-1}(\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,\mathrm{SL}(n,\mathbb{C})})$.
(II) Let $b_i\in C^{2i-1}(G\times_G BG)$ be a singular cochain complex ($G=\mathrm{SL}(n,\mathbb{C})$) such that $j^*(b_i)=0$, where j is the inclusion $BG\to G\times_G EG$ as $\{e\}\times_G EG$, and generate the cohomology of the fibar G of filtration $G\times_G EG\to G$ as a ring. For

$$\Phi^G:G^{2g}\times\prod_l\mathcal{C}_l\to G$$
$$(A_1,B_1,\dots,A_g,B_g,X_1,\dots,X_k)\mapsto[A_1,B_1]\cdots[A_g,B_g]X_1\cdots X_k,$$

we can show that $\Phi^{G*}(b_i)=0$ in $H^{2i-1}_G(G^{2g}\times\prod_l\mathcal{C}_l)$. We take $a_i\in C^{2i-2}((G^{2g}\times\prod_l\mathcal{C}_l)\times_G EG)$ such that $da_i=\Phi^{G*}(b_i)$. Then,

$$\alpha_i:=[a_i|_{\mathcal{U}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_G}]\in H^{2i-2}_G(\mathcal{U}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_G)\cong H^{2i-2}(\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,G}).$$

(III) We fix diagonal matrices $D_1,\cdots,D_k$ in each conjugacy classes $\mathcal{C}_1,\dots,\mathcal{C}_k$.

$$\mathrm{GL}(n,\mathbb{C})/H_l\to\mathcal{C}_l;\quad[g]\mapsto g^{-1}D_lg,$$

where $H_l=\mathrm{GL}(\mu^l_1,\mathbb{C})\oplus\cdots\oplus\mathrm{GL}(\mu^l_{r_l},\mathbb{C})$. ($\mu^l_1+\cdots+\mu^l_{r_l}=n$: the multiplicities of the eigenvalues of any matrix in $\mathcal{C}_l$). For a conjugacy class $\mathcal{C}_l$,

$$'\mathcal{U}^{\boldsymbol{\mu},\boldsymbol{\lambda},g,l}_{\mathrm{SL}(n,\mathbb{C})}:=\{(A_1,B_1,\dots,A_g,B_g;X_1,\dots,M_l,\dots,X_k)$$
$$\in\mathrm{SL}(n,\mathbb{C})^{2g}\times\Pi^{l-1}_{j=1}\mathcal{C}_j\times\mathrm{GL}(n,\mathbb{C})\times\Pi^k_{j=l+1}\mathcal{C}_j$$
$$\mid\prod_{i=1}^g(A_i,B_i)X_1\cdots M_l^{-1}D_lM_l\cdots X_k=I_n\}.$$

A natural map

$$'\mathcal{U}^{\boldsymbol{\mu},\boldsymbol{\lambda},g,l}_{\mathrm{SL}(n,\mathbb{C})}\to\mathcal{U}^{\boldsymbol{\mu},\boldsymbol{\lambda},k}_{\mathrm{SL}(n,\mathbb{C})};\quad(\dots,M_i,\dots)\mapsto(\dots,M_l^{-1}D_lM_l,\dots).$$

We put $'\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g,l}_{B,\mathrm{SL}(n,\mathbb{C})}:={}'\mathcal{U}^{\boldsymbol{\mu},\boldsymbol{\lambda},g,l}_{\mathrm{SL}(n,\mathbb{C})}//\mathrm{PGL}(n,\mathbb{C})$ by a natural $\mathrm{PGL}(n,\mathbb{C})$-action, which is a H_l-principal bundle on $\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,\mathrm{SL}(n,\mathbb{C})}$. we consider the classifying map,

$$\begin{array}{ccl} '\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g,l}_{B,\mathrm{SL}(n,\mathbb{C})} & & \\ \downarrow & & \\ \mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,\mathrm{SL}(n,\mathbb{C})} & \xrightarrow{f} & BH_l\cong B\mathrm{GL}(\mu^l_1,\mathbb{C})\times\cdots\times B\mathrm{GL}(\mu^l_{r_l},\mathbb{C}). \end{array}$$

The characteristic classes

$$\epsilon^l_{k_1,\dots,k_{r_l}}:=f^*(c_{k_1}\otimes\cdots\otimes c_{k_{r_l}})\quad(0\leq k_j\leq\mu^l_j),$$

where $c_{k_j}\in H^{2k_j}(B\mathrm{GL}(\mu^l_j,\mathbb{C}))$ and $c_0:=1$.

Theorem 3

The classes $\alpha_i,\beta_i,\gamma_{i,j}$, and $\epsilon^l_{k_1,\dots,k_{r_l}}$ generate $H^*(\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},g}_{B,\mathrm{SL}(n,\mathbb{C})})^{\nu_n^{2g}}$.

The generators α_i, β_i, $\gamma_{i,j}$ (resp. $\epsilon^l_{k_1,\dots,k_{r_l}}$) have homogeneous weight i (resp. $k_1+\cdots+k_{r_l}$). On the other hand, generators of $H^*((\mathbb{C}^\times)^{2g})$ have also homogeneous weight. Then,

Theorem 4

The cohomology of $\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},k}_{B,\mathrm{GL}(n,\mathbb{C})}$ is type (p,p)
i.e. $\dim_{\mathbb{C}}\mathrm{Gr}^p_F\mathrm{Gr}^W_{p+q}H^j(\mathcal{M}^{\boldsymbol{\mu},\boldsymbol{\lambda},k}_{B,\mathrm{GL}(n,\mathbb{C})})=0$ unless $p=q$.

References

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